13-4 Double-Angle and Half-Angle Identities

CCSS PRECISION Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$.

1. $\sin\theta = \frac{1}{4};\ 0° < \theta < 90°$

SOLUTION:

We know that $\cos^2\theta = 1 - \sin^2\theta$.

Substitute $\frac{1}{4}$ for $\sin\theta$.

$$\begin{aligned}\cos^2\theta &= 1 - \left(\frac{1}{4}\right)^2 \\ &= 1 - \frac{1}{16} \\ &= \frac{15}{16} \\ \cos\theta &= \pm\frac{\sqrt{15}}{4}\end{aligned}$$

Since $0° < \theta < 90°$, cosine is positive.

Thus, $\cos\theta = \frac{\sqrt{15}}{4}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{1}{4}\right)\left(\frac{\sqrt{15}}{4}\right) \\ &= \frac{\sqrt{15}}{8}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 1 - 2\sin^2\theta \\ &= 1 - 2\left(\frac{1}{4}\right)^2 \\ &= 1 - 2\left(\frac{1}{16}\right) \\ &= \frac{7}{8}\end{aligned}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$= \sqrt{\frac{1-\frac{\sqrt{15}}{4}}{2}}, \text{ where } 0° < \frac{\theta}{2} < 45°$$
$$= \sqrt{\frac{4-\sqrt{15}}{8}}$$
$$= \frac{\sqrt{4-\sqrt{15}}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{8-2\sqrt{15}}}{4}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$= \sqrt{\frac{1+\frac{\sqrt{15}}{4}}{2}}, \text{ where } 0° < \frac{\theta}{2} < 45°$$
$$= \sqrt{\frac{4+\sqrt{15}}{8}}$$
$$= \frac{\sqrt{4+\sqrt{15}}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{8+2\sqrt{15}}}{4}$$

2. $\sin\theta = \frac{4}{5};\ 90° < \theta < 180°$

SOLUTION:

We know that

$\cos^2\theta = 1 - \sin^2\theta$.

Substitute $\frac{4}{5}$ for $\sin\theta$.

$$\cos^2\theta = 1 - \left(\frac{4}{5}\right)^2$$
$$= 1 - \frac{16}{25}$$
$$= \frac{9}{25}$$
$$\cos\theta = \pm\frac{3}{5}$$

Since $90° < \theta < 180°$, $\cos\theta = -\frac{3}{5}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) \\ &= -\frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 1-2\sin^2\theta \\ &= 1-2\left(\frac{4}{5}\right)^2 \\ &= 1-2\left(\frac{16}{25}\right) \\ &= -\frac{7}{25}\end{aligned}$$

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\ &= \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}, \text{where } 45° < \frac{\theta}{2} < 90° \\ &= \sqrt{\frac{1+\frac{3}{5}}{2}} \\ &= \frac{2\sqrt{2}}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} \\ &= \frac{4\sqrt{5}}{10} \\ &= \frac{2\sqrt{5}}{5}\end{aligned}$$

$$\begin{aligned}\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}\\ &= \sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}}, \text{ where } 45° < \frac{\theta}{2} < 90°\\ &= \frac{\sqrt{2}}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}\\ &= \frac{2\sqrt{5}}{10}\\ &= \frac{\sqrt{5}}{5}\end{aligned}$$

3. $\cos\theta = -\frac{5}{13}; \frac{\pi}{2} < \theta < \pi$

SOLUTION:

We know that

$\sin^2\theta = 1 - \cos^2\theta$.

Substitute $-\frac{5}{13}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta &= 1 - \left(-\frac{5}{13}\right)^2\\ &= 1 - \frac{25}{169}\\ &= \frac{144}{169}\\ \sin\theta &= \pm\frac{12}{13}\end{aligned}$$

Since $\frac{\pi}{2} < \theta < \pi$, $\sin\theta = \frac{12}{13}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{12}{13}\right)\left(-\frac{5}{13}\right)\\ &= -\frac{120}{169}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{5}{13}\right)^2 - 1\\ &= 2\left(\frac{25}{169}\right) - 1\\ &= \frac{50}{169} - 1\\ &= -\frac{119}{169}\end{aligned}$$

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}}\\ &= \sqrt{\frac{1-\left(-\frac{5}{13}\right)}{2}}, \text{where } \frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}\\ &= \sqrt{\frac{1+\frac{5}{13}}{2}}\\ &= \frac{\sqrt{18}}{\sqrt{26}}\\ &= \frac{3\sqrt{2}}{\sqrt{13}\sqrt{2}} \times \frac{\sqrt{13}\sqrt{2}}{\sqrt{13}\sqrt{2}}\\ &= \frac{3\sqrt{13}}{13}\end{aligned}$$

$$\begin{aligned}\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}\\ &= \sqrt{\frac{1+\left(-\frac{5}{13}\right)}{2}}, \text{where } \frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}\\ &= \sqrt{\frac{\frac{8}{13}}{2}}\\ &= \frac{2\sqrt{2}}{\sqrt{13}\sqrt{2}} \times \frac{\sqrt{13}\sqrt{2}}{\sqrt{13}\sqrt{2}}\\ &= \frac{2\sqrt{13}}{13}\end{aligned}$$

4. $\cos\theta = \frac{3}{5}; 270° < \theta < 360°$

SOLUTION:

We know that

$\sin^2\theta = 1 - \cos^2\theta$.

Substitute $\frac{3}{5}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta &= 1 - \left(\frac{3}{5}\right)^2\\ &= 1 - \frac{9}{25}\\ &= \frac{16}{25}\\ \sin\theta &= \pm\frac{4}{5}\end{aligned}$$

Since $270° < \theta < 360°$, $\sin\theta = -\frac{4}{5}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{3}{5}\right)^2 - 1\\ &= -\frac{7}{25}\end{aligned}$$

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}}\\ &= \sqrt{\frac{1-\frac{3}{5}}{2}},\text{ where } 135° < \frac{\theta}{2} < 180°\\ &= \sqrt{\frac{\frac{2}{5}}{2}}\\ &= \frac{\sqrt{2}}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}\\ &= \frac{\sqrt{5}}{5}\end{aligned}$$

$$\begin{aligned}\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}\\ &= -\sqrt{\frac{1+\frac{3}{5}}{2}},\text{ where }135^\circ < \frac{\theta}{2} < 180^\circ\\ &= -\sqrt{\frac{\frac{8}{5}}{2}}\\ &= -\frac{2\sqrt{2}}{\sqrt{10}}\times\frac{\sqrt{10}}{\sqrt{10}}\\ &= -\frac{2\sqrt{5}}{5}\end{aligned}$$

5. $\tan\theta = \frac{8}{15}; 90^\circ < \theta < 180^\circ$

SOLUTION:

We know that

$\tan^2\theta + 1 = \sec^2\theta$.

Substitute $\frac{8}{15}$ for $\tan\theta$.

$$\begin{aligned}\left(\frac{8}{15}\right)^2 + 1 &= \sec^2\theta\\ \frac{64}{225} + 1 &= \sec^2\theta\\ \sec\theta &= \pm\frac{17}{15}\\ \cos\theta &= \pm\frac{15}{17}\end{aligned}$$

Since $90^\circ < \theta < 180^\circ$, $\cos\theta = -\frac{15}{17}$.

We know that

$\sin^2\theta = 1 - \cos^2\theta$.

Substitute $-\frac{15}{17}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta &= 1-\left(-\frac{15}{17}\right)^2\\ &= 1-\frac{225}{289}\\ &= \frac{64}{289}\\ \sin\theta &= \pm\frac{8}{17}\end{aligned}$$

Since $90° < \theta < 180°$, $\sin\theta = \frac{8}{17}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{8}{17}\right)\left(-\frac{15}{17}\right)\\ &= -\frac{240}{289}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{15}{17}\right)^2 - 1\\ &= 2\left(\frac{225}{289}\right) - 1\\ &= \frac{161}{289}\end{aligned}$$

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}}\\ &= \sqrt{\frac{1-\left(-\frac{15}{17}\right)}{2}},\ \text{where } 45° < \frac{\theta}{2} < 90°\\ &= \sqrt{\frac{1+\frac{15}{17}}{2}}\\ &= \frac{4\sqrt{2}}{\sqrt{17}\sqrt{2}} \times \frac{\sqrt{17}\sqrt{2}}{\sqrt{17}\sqrt{2}}\\ &= \frac{4\sqrt{17}}{17}\end{aligned}$$

$$\begin{aligned}\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}\\ &= \sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}},\text{ where } 45° < \frac{\theta}{2} < 90°\\ &= \sqrt{\frac{2}{34}}\\ &= \frac{\sqrt{2}}{\sqrt{17}\sqrt{2}} \times \frac{\sqrt{17}\sqrt{2}}{\sqrt{17}\sqrt{2}}\\ &= \frac{\sqrt{17}}{17}\end{aligned}$$

6. $\tan\theta = \frac{5}{12}; \pi < \theta < \frac{3\pi}{2}$

SOLUTION:

We know that

$\tan^2\theta + 1 = \sec^2\theta$.

Substitute $\frac{5}{12}$ for $\tan\theta$.

$$\begin{aligned}\left(\frac{5}{12}\right)^2 + 1 &= \sec^2\theta\\ \frac{25}{144} + 1 &= \sec^2\theta\\ \frac{169}{144} &= \sec^2\theta\\ \sec\theta &= \pm\frac{13}{12}\\ \cos\theta &= \pm\frac{12}{13}\end{aligned}$$

Since $\pi < \theta < \frac{3\pi}{2}$, $\cos\theta = -\frac{12}{13}$.

We know that

$\sin^2\theta = 1 - \cos^2\theta$.

Substitute $-\frac{12}{13}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta &= 1-\left(-\frac{12}{13}\right)^2 \\ &= 1-\frac{144}{169} \\ &= \frac{25}{169} \\ \sin\theta &= \pm\sqrt{\frac{25}{169}} \\ &= \pm\frac{5}{13}\end{aligned}$$

Since $\pi < \theta < \frac{3\pi}{2}$, $\sin\theta = -\frac{5}{13}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(-\frac{5}{13}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{120}{169}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(-\frac{12}{13}\right)^2 - 1 \\ &= 2\left(\frac{144}{169}\right) - 1 \\ &= \frac{288}{169} - 1 \\ &= \frac{119}{169}\end{aligned}$$

$$\begin{aligned}
\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
&= \sqrt{\frac{1-\left(-\frac{12}{13}\right)}{2}}, \text{ where } \frac{\pi}{2}<\frac{\theta}{2}<\frac{3\pi}{4} \\
&= \sqrt{\frac{1+\frac{12}{13}}{2}} \\
&= \sqrt{\frac{25}{26}} \\
&= \frac{5}{\sqrt{26}} \\
&= \frac{5}{\sqrt{26}} \times \frac{\sqrt{26}}{\sqrt{26}} \\
&= \frac{5\sqrt{26}}{26}
\end{aligned}$$

$$\begin{aligned}
\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} \\
&= -\sqrt{\frac{1+\left(-\frac{12}{13}\right)}{2}}, \text{ where } \frac{\pi}{2}<\frac{\theta}{2}<\frac{3\pi}{4} \\
&= -\sqrt{\frac{1-\frac{12}{13}}{2}} \\
&= -\sqrt{\frac{1}{26}} \\
&= -\frac{1}{\sqrt{26}} \\
&= -\frac{1}{\sqrt{26}} \times \frac{\sqrt{26}}{\sqrt{26}} \\
&= -\frac{\sqrt{26}}{26}
\end{aligned}$$

13-4 Double-Angle and Half-Angle Identities

Find the exact value of each expression.

7. $\sin \frac{\pi}{8}$

SOLUTION:

$$\begin{aligned}\sin \frac{\pi}{8} &= \sin \frac{180^\circ}{8} \\ &= \sin 22.5^\circ \\ &= \sin \frac{45^\circ}{2}\end{aligned}$$

We know that

$$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\sin \frac{45^\circ}{2} = \pm\sqrt{\frac{1-\cos 45^\circ}{2}}$$

Since 45° is in first quadrant, sine is positive.

$$\begin{aligned}\sin \frac{45^\circ}{2} &= \sqrt{\frac{1-\cos 45^\circ}{2}} \\ &= \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} \\ &= \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}} \\ &= \sqrt{\frac{2-\sqrt{2}}{4}} \\ &= \frac{\sqrt{2-\sqrt{2}}}{2}\end{aligned}$$

8. $\cos 15°$

SOLUTION:

$$\cos 15° = \cos \frac{30°}{2}$$

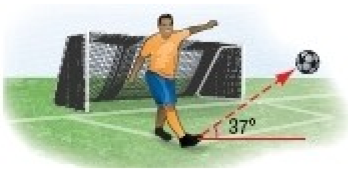
We know that

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$
$$\cos \frac{30°}{2} = \pm \sqrt{\frac{1 + \cos 30°}{2}}$$

Since $30°$ is in first quadrant, the cosine value is positive.

$$\begin{aligned} \cos \frac{30°}{2} &= \sqrt{\frac{1 + \cos 30°}{2}} \\ &= \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{2 + \sqrt{3}}{4}} \\ &= \frac{\sqrt{2 + \sqrt{3}}}{2} \end{aligned}$$

9. **SOCCER** A soccer player kicks a ball at an angle of 37° with the ground with an initial velocity of 52 feet per second. The distance *d* that the ball will go in the air if it is not blocked is given by $d = \frac{2v^2 \sin\theta\cos\theta}{g}$. In this formula, *g* is the acceleration due to gravity and is equal to 32 feet per second squared, and *v* is the initial velocity.
a. Simplify this formula by using a double-angle identity.
b. Using the simplified formula, how far will this ball go?



SOLUTION:
a.

$$\begin{aligned} d &= \frac{2v^2 \sin\theta\cos\theta}{g} \\ &= \frac{v^2 2\sin\theta\cos\theta}{g} \end{aligned}$$

The double angle identity is:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Replace $2\sin\theta\cos\theta$ by $\sin 2\theta$ in the equation. $d = \frac{v^2 2\sin\theta\cos\theta}{g}$.

$$d = \frac{v^2 \sin 2\theta}{g}$$

b. Substitute $37°$ for θ, 52 for *v*, and 32 for *g*.

$$\begin{aligned} d &= \frac{(52)^2 \sin 2(37°)}{32} \\ &= \frac{2704\sin(74°)}{32} \\ &\approx 81 \end{aligned}$$

The ball will go approximately 81 feet.

13-4 Double-Angle and Half-Angle Identities

Verify that each equation is an identity.

10. $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$

SOLUTION:

$$\begin{aligned}
\tan\theta &\stackrel{?}{=} \frac{1-\cos 2\theta}{\sin 2\theta}\\
&\stackrel{?}{=} \frac{1-\left(1-2\sin^2\theta\right)}{2\sin\theta\cos\theta}\\
&\stackrel{?}{=} \frac{1-1+2\sin^2\theta}{2\sin\theta\cos\theta}\\
&\stackrel{?}{=} \frac{2\sin^2\theta}{2\sin\theta\cos\theta}\\
&\stackrel{?}{=} \frac{\sin\theta}{\cos\theta}\\
&= \tan\theta \checkmark
\end{aligned}$$

11. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

SOLUTION:

$$\begin{aligned}
(\sin\theta+\cos\theta)^2 &\stackrel{?}{=} 1+2\sin\theta\cos\theta\\
(\sin\theta+\cos\theta)(\sin\theta+\cos\theta) &\stackrel{?}{=} 1+2\sin\theta\cos\theta\\
\sin^2\theta+\sin\theta\cos\theta+\sin\theta\cos\theta+\cos^2\theta &\stackrel{?}{=} 1+2\sin\theta\cos\theta\\
\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta &\stackrel{?}{=} 1+2\sin\theta\cos\theta\\
1+2\sin\theta\cos\theta &= 1+2\sin\theta\cos\theta \checkmark
\end{aligned}$$

Find the exact values of sin 2θ, $\cos 2\theta$, $\sin\dfrac{\theta}{2}$ and $\cos\dfrac{\theta}{2}$.

12. $\sin\theta = \dfrac{2}{3}; 90° < \theta < 180°$

SOLUTION:

We know that

$\cos^2\theta = 1 - \sin^2\theta$.

Substitute $\dfrac{2}{3}$ for $\sin\theta$.

$$\cos^2\theta = 1 - \left(\frac{2}{3}\right)^2$$
$$= 1 - \frac{4}{9}$$
$$= \frac{5}{9}$$
$$\cos\theta = \pm\frac{\sqrt{5}}{3}$$

Since $90° < \theta < 180°$, $\cos\theta = -\frac{\sqrt{5}}{3}$.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)$$
$$= -\frac{4\sqrt{5}}{9}$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$= 1 - 2\left(\frac{2}{3}\right)^2$$
$$= 1 - 2\left(\frac{4}{9}\right)$$
$$= \frac{1}{9}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$= \sqrt{\frac{1-\left(-\frac{\sqrt{5}}{3}\right)}{2}}, \text{where } 45° < \frac{\theta}{2} < 90°$$
$$= \sqrt{\frac{\frac{3+\sqrt{5}}{3}}{2}}$$
$$= \frac{\sqrt{3+\sqrt{5}}}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$
$$= \frac{\sqrt{6}\cdot\sqrt{3+\sqrt{5}}}{6}$$

$$\begin{aligned}\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}\\ &= \sqrt{\frac{1+\left(-\frac{\sqrt{5}}{3}\right)}{2}}, \text{ where } 45° < \frac{\theta}{2} < 90°\\ &= \sqrt{\frac{\frac{3-\sqrt{5}}{3}}{2}}\\ &= \frac{\sqrt{3-\sqrt{5}}}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}\\ &= \frac{\sqrt{6}\cdot\sqrt{3-\sqrt{5}}}{6}\end{aligned}$$

13. $\sin\theta = -\frac{15}{17};\ \pi < \theta < \frac{3\pi}{2}$

SOLUTION:

We know that

$\cos^2\theta = 1 - \sin^2\theta$.

Substitute $-\frac{15}{17}$ for $\sin\theta$.

$$\begin{aligned}\cos^2\theta &= 1-\left(-\frac{15}{17}\right)^2\\ &= 1-\frac{225}{289}\\ &= \frac{64}{289}\\ \cos\theta &= \pm\frac{8}{17}\end{aligned}$$

Since, $\pi < \theta < \frac{3\pi}{2}$, $\cos\theta = -\frac{8}{17}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{15}{17}\right)\left(-\frac{8}{17}\right)\\ &= \frac{240}{289}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 1 - 2\sin^2\theta \\ &= 1 - 2\left(-\frac{15}{17}\right)^2 \\ &= 1 - 2\left(\frac{225}{289}\right) \\ &= 1 - \frac{450}{289} \\ &= -\frac{161}{289}\end{aligned}$$

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\ &= \sqrt{\frac{1-\left(-\frac{8}{17}\right)}{2}}, \text{where } \frac{\pi}{2} < \theta < \frac{3\pi}{4} \\ &= \sqrt{\frac{1+\frac{8}{17}}{2}} \\ &= \sqrt{\frac{25}{34}} \\ &= \frac{5}{\sqrt{34}} \times \frac{\sqrt{34}}{\sqrt{34}} \\ &= \frac{5\sqrt{34}}{34}\end{aligned}$$

$$\begin{aligned}\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} \\ &= \sqrt{\frac{1+\left(-\frac{8}{17}\right)}{2}}, \text{where } \frac{\pi}{2} < \theta < \frac{3\pi}{4} \\ &= \sqrt{\frac{-9}{34}} \\ &= -\frac{3}{\sqrt{34}} \times \frac{\sqrt{34}}{\sqrt{34}} \\ &= -\frac{3\sqrt{34}}{34}\end{aligned}$$

14. $\cos\theta = \frac{3}{5};\ \frac{3\pi}{2} < \theta < 2\pi$

SOLUTION:

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\sin^2\theta = 1 - \left(\frac{3}{5}\right)^2$$
$$\sin^2\theta = \frac{16}{25}$$
$$\sin\theta = \frac{4}{5}$$

Since θ is between $\frac{3\pi}{2}$ and 2π, $\sin\theta$ is negative. Therefore, $\sin\theta = -\frac{4}{5}$.

Find $\sin 2\theta$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) \\ &= -\frac{24}{25}\end{aligned}$$

Find $\cos 2\theta$.

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 \\ &= \frac{9}{25} - \frac{16}{25} \\ &= -\frac{7}{25}\end{aligned}$$

Find $\sin\frac{\theta}{2}$.

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}}\\ &= \pm\sqrt{\frac{1-\frac{3}{5}}{2}}\\ &= \pm\sqrt{\frac{\frac{5}{5}-\frac{3}{5}}{2}}\\ &= \pm\sqrt{\frac{\frac{2}{5}}{2}}\\ &= \pm\sqrt{\frac{2}{10}}\\ &= \pm\frac{\sqrt{2}}{\sqrt{10}}\\ &= \pm\frac{\sqrt{20}}{10}\\ &= \pm\frac{2\sqrt{5}}{10}\\ &= \pm\frac{\sqrt{5}}{5}\end{aligned}$$

Because θ is between 270º and 360º, $\frac{\theta}{2}$ is between 135º and 180º. Therefore, $\sin\frac{\theta}{2} = \frac{\sqrt{5}}{5}$.

Find $\cos\frac{\theta}{2}$.

$$\begin{aligned}\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}\\ &= \pm\sqrt{\frac{1+\frac{3}{5}}{2}}\\ &= \pm\sqrt{\frac{\frac{5}{5}+\frac{3}{5}}{2}}\\ &= \pm\sqrt{\frac{\frac{8}{5}}{2}}\\ &= \pm\sqrt{\frac{8}{10}}\\ &= \pm\frac{\sqrt{8}}{\sqrt{10}}\\ &= \pm\frac{\sqrt{80}}{10}\\ &= \pm\frac{4\sqrt{5}}{10}\\ &= \pm\frac{2\sqrt{5}}{5}\end{aligned}$$

Because θ is between 270º and 360º, $\frac{\theta}{2}$ is between 135º and 180º. Therefore, $\cos\frac{\theta}{2} = -\frac{2\sqrt{5}}{5}$.

15. $\cos\theta = \frac{1}{5};\ 270° < \theta < 360°$

SOLUTION:

We know that $\sin^2\theta = 1-\cos^2\theta$.

Substitute $\frac{1}{5}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta &= 1-\left(\frac{1}{5}\right)^2\\ &= 1-\frac{1}{25}\\ &= \frac{24}{25}\\ \sin\theta &= \pm\frac{2\sqrt{6}}{5}\end{aligned}$$

Since $270° < \theta < 360°$, $\sin\theta = -\frac{2\sqrt{6}}{5}$.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(-\frac{2\sqrt{6}}{5}\right)\left(\frac{1}{5}\right)$$
$$= -\frac{4\sqrt{6}}{25}$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$= 2\left(\frac{1}{5}\right)^2 - 1$$
$$= 2\left(\frac{1}{25}\right) - 1$$
$$= -\frac{23}{25}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$= \sqrt{\frac{1-\frac{1}{5}}{2}}, \text{ where } 135° < \theta < 180°$$
$$= \sqrt{\frac{\frac{4}{5}}{2}}$$
$$= \frac{2}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$$
$$= \frac{2\sqrt{10}}{10}$$
$$= \frac{\sqrt{10}}{5}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$= -\sqrt{\frac{1+\frac{1}{5}}{2}}, \text{ where } 135° < \theta < 180°$$
$$= -\sqrt{\frac{\frac{6}{5}}{2}}$$
$$= -\frac{\sqrt{6}}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$$
$$= -\frac{\sqrt{15}}{5}$$

13-4 Double-Angle and Half-Angle Identities

16. $\tan\theta = \frac{4}{3};\ 180° < \theta < 270°$

SOLUTION:

We know that $\tan^2\theta + 1 = \sec^2\theta$.

Substitute $\frac{4}{3}$ for $\tan\theta$.

$$\left(\frac{4}{3}\right)^2 + 1 = \sec^2\theta$$
$$\frac{16}{9} + 1 = \sec^2\theta$$
$$\frac{25}{9} = \sec^2\theta$$
$$\sec\theta = \pm\frac{5}{3}$$
$$\cos\theta = \pm\frac{3}{5}$$

Since $180° < \theta < 270°$, $\cos\theta = -\frac{3}{5}$.

We know that,

$\sin^2\theta = 1 - \cos^2\theta$.

Substitute $-\frac{3}{5}$ for $\cos\theta$.

$$\sin^2\theta = 1 - \left(-\frac{3}{5}\right)^2$$
$$= 1 - \frac{9}{25}$$
$$= \frac{16}{25}$$
$$\sin\theta = \pm\frac{4}{5}$$

Since, $180° < \theta < 270°$, $\sin\theta = -\frac{4}{5}$.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right)$$
$$= \frac{24}{25}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{3}{5}\right)^2 - 1\\ &= 2\left(\frac{9}{25}\right) - 1\\ &= \frac{18}{25} - 1\\ &= -\frac{7}{25}\end{aligned}$$

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}}\\ &= \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}, \text{ where } 90° < \theta < 135°\\ &= \sqrt{\frac{1+\frac{3}{5}}{2}}\\ &= \frac{2\sqrt{2}}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}\\ &= \frac{2\sqrt{20}}{10}\\ &= \frac{2\sqrt{5}}{5}\end{aligned}$$

$$\begin{aligned}\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}\\ &= -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}}, \text{ where } 90° < \theta < 135°\\ &= -\sqrt{\frac{1-\frac{3}{5}}{2}}\\ &= -\frac{\sqrt{2}}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}\\ &= -\frac{\sqrt{20}}{10}\\ &= -\frac{\sqrt{5}}{5}\end{aligned}$$

17. $\tan\theta = -2;\ \frac{\pi}{2} < \theta < \pi$

SOLUTION:

We know that

$\tan^2\theta + 1 = \sec^2\theta$.

Substitute –2 for $\tan\theta$.

$$\begin{aligned}(-2)^2 + 1 &= \sec^2\theta\\ 4 + 1 &= \sec^2\theta\\ 5 &= \sec^2\theta\\ \sec\theta &= \pm\sqrt{5}\\ \cos\theta &= \pm\frac{1}{\sqrt{5}}\end{aligned}$$

Since, $\frac{\pi}{2} < \theta < \pi$, $\cos\theta = -\frac{1}{\sqrt{5}}$.

We know that,

$\sin^2\theta = 1 - \cos^2\theta$.

Substitute $-\frac{1}{\sqrt{5}}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta &= 1 - \left(-\frac{1}{\sqrt{5}}\right)^2\\ &= 1 - \frac{1}{5}\\ &= \frac{4}{5}\\ \sin\theta &= \pm\frac{2}{\sqrt{5}}\end{aligned}$$

Since $\frac{\pi}{2} < \theta < \pi$, $\sin\theta = \frac{2}{\sqrt{5}}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{\sqrt{5}}\right)\\ &= -\frac{4}{5}\end{aligned}$$

13-4 Double-Angle and Half-Angle Identities

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{1}{\sqrt{5}}\right)^2 - 1\\ &= 2\left(\frac{1}{5}\right) - 1\\ &= -\frac{3}{5}\end{aligned}$$

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}}\\ &= \sqrt{\frac{1-\left(-\frac{1}{\sqrt{5}}\right)}{2}}, \text{ where } \frac{\pi}{4} < \theta < \frac{\pi}{2}\\ &= \sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}}\\ &= \sqrt{\frac{\frac{\sqrt{5}+1}{\sqrt{5}}}{2}}\\ &= \sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}\end{aligned}$$

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}\\ &= \sqrt{\frac{1+\left(-\frac{1}{\sqrt{5}}\right)}{2}}, \text{ where } \frac{\pi}{4} < \theta < \frac{\pi}{2}\\ &= \sqrt{\frac{1-\frac{1}{\sqrt{5}}}{2}}\\ &= \sqrt{\frac{\frac{\sqrt{5}-1}{\sqrt{5}}}{2}}\\ &= \sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}\end{aligned}$$

Find the exact value of each expression.

18. $\sin 75°$

SOLUTION:

$$\sin 75° = \sin \frac{150°}{2}$$

Use the half-angle sine formula.

$$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin \frac{150°}{2} = \pm\sqrt{\frac{1-\cos 150°}{2}}$$

$$\begin{aligned}\cos 150° &= \cos(90° + 60°)\\ &= -\sin 60°\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Substitute $-\frac{\sqrt{3}}{2}$ for $\cos 150°$.

$$\begin{aligned}\sin \frac{150°}{2} &= \pm\sqrt{\frac{1-\cos 150°}{2}}\\ &= \pm\sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}}\\ &= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}\\ &= \pm\frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

Since θ lies in the second quadrant, sine is positive.

$$\sin 75° = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

19. $\sin\frac{3\pi}{8}$

SOLUTION:

$$\sin\frac{3\pi}{8} = \sin 67.5° = \sin\frac{135°}{2}$$

Use the half-angle sine formula.

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\sin\frac{135°}{2} = \pm\sqrt{\frac{1-\cos 135°}{2}}$$

$$\begin{aligned}\cos 135° &= \cos(90° + 45°)\\ &= -\sin 45°\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

Substitute $-\frac{1}{\sqrt{2}}$ for $\cos 135°$.

$$\begin{aligned}\sin\frac{135°}{2} &= \pm\sqrt{\frac{1-\cos 135°}{2}}\\ &= \pm\sqrt{\frac{1-\left(-\frac{1}{\sqrt{2}}\right)}{2}}\\ &= \pm\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}\\ &= \pm\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}}\\ &= \pm\sqrt{\frac{2+\sqrt{2}}{4}}\\ &= \pm\frac{\sqrt{2+\sqrt{2}}}{2}\end{aligned}$$

Since $\theta = 135°$ is lies in the second quadrant, sine is positive.

$$\sin\frac{3\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

20. $\cos\frac{7\pi}{12}$

SOLUTION:

$$\cos\frac{7\pi}{12} = \cos 75° = \cos\frac{150°}{2}$$

Use the half-angle cosine formula.

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{150°}{2} = \pm\sqrt{\frac{1+\cos 150°}{2}}$$

$$\begin{aligned}\cos 150° &= \cos(90° + 60°)\\ &= -\sin 60°\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Substitute $-\frac{\sqrt{3}}{2}$ for $\cos 150°$.

$$\begin{aligned}\cos\frac{150°}{2} &= \pm\sqrt{\frac{1+\cos 150°}{2}}\\ &= \pm\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}}\\ &= \pm\sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \pm\frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Since θ lies in the second quadrant, cosine is negative.

$$\cos\frac{150°}{2} = -\frac{\sqrt{2-\sqrt{3}}}{2}.$$

21. $\tan 165°$

SOLUTION:

$$\tan 165° = \tan(120° + 45°)$$
$$= \frac{\tan 120° + \tan 45°}{1 - \tan 120° \tan 45°}$$

$$\tan 120° = \tan(90° + 30°)$$
$$= -\cot(30°)$$
$$= -\sqrt{3}$$

$$\frac{\tan 120° + \tan 45°}{1 - \tan 120° \tan 45°} = \frac{-\sqrt{3} + 1}{1 - \left(-\sqrt{3}\right) \cdot 1}$$
$$= \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}}$$
$$= \frac{1 - 2\sqrt{3} + 3}{1 - 3}$$
$$= \frac{4 - 2\sqrt{3}}{-2}$$
$$= \sqrt{3} - 2$$

22. $\tan\frac{5\pi}{12}$

SOLUTION:

$$\tan\frac{5\pi}{12}=\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$$

Use the following formula.

$$\begin{aligned}
\tan(A+B)&=\frac{\tan A+\tan B}{1-\tan A\tan B}\\
\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)&=\frac{\tan\frac{\pi}{6}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{6}\tan\frac{\pi}{4}}\\
&=\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}\cdot 1}\\
&=\frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}\\
&=\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\\
&=\frac{\left(\sqrt{3}^2\right)+2\sqrt{3}+1^2}{\left(\sqrt{3}^2\right)-1^2}\\
&=\frac{4+2\sqrt{3}}{2}\\
&=2+\sqrt{3}
\end{aligned}$$

23. $\tan 22.5°$

SOLUTION:

$$\tan 22.5° = \tan\frac{45°}{2}$$

Use the half-angle tangent formula.

$$\begin{aligned}
\tan\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\\
\tan\frac{45°}{2} &= \pm\sqrt{\frac{1-\cos 45°}{1+\cos 45°}} \text{ where } \theta \text{ lies in first quadrant}\\
&= \sqrt{\frac{1-\cos 45°}{1+\cos 45°}}\\
&= \frac{\sqrt{1-\cos 45°}}{\sqrt{1+\cos 45°}}\cdot\frac{\sqrt{1+\cos 45°}}{\sqrt{1+\cos 45°}}\\
&= \frac{\sqrt{(1-\cos 45°)(1+\cos 45°)}}{1+\cos 45°}\\
&= \frac{\sqrt{1-\cos^2 45°}}{1+\cos 45°}\\
&= \frac{\sin 45°}{1+\cos 45°}\\
&= \frac{\left(\frac{1}{\sqrt{2}}\right)}{1+\frac{1}{\sqrt{2}}}\\
&= \frac{1}{\sqrt{2}+1}\cdot\frac{1-\sqrt{2}}{1-\sqrt{2}}\\
&= \frac{1-\sqrt{2}}{1-2}\\
&= \sqrt{2}-1
\end{aligned}$$

24. **GEOGRAPHY** The Mercator projection of the globe is a projection on which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection involves the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.

a. Write this expression in terms of a trigonometric function of L.

b. The latitude of Tallahassee, Florida, is 30° north. Find the value of the expression if L = 30°.

SOLUTION:

a. $\tan\left(45° + \frac{L}{2}\right) = \dfrac{\tan 45° + \tan\left(\frac{L}{2}\right)}{1 - \tan 45° \tan\left(\frac{L}{2}\right)}$

Use the formula $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ for $\tan\frac{L}{2}$.

$$\tan\left(45° + \frac{L}{2}\right) = \frac{1 \pm \sqrt{\frac{1-\cos L}{1+\cos L}}}{1 - 1 \cdot \left(\pm\sqrt{\frac{1-\cos L}{1+\cos L}}\right)}$$
$$= \frac{1 \pm \sqrt{\frac{1-\cos L}{1+\cos L}}}{1 \pm \sqrt{\frac{1-\cos L}{1+\cos L}}}$$

b. Substitute 30° for *L*.

$$\frac{1+\sqrt{\frac{1-\cos 30°}{1+\cos 30°}}}{1-\sqrt{\frac{1-\cos 30°}{1+\cos 30°}}} = \frac{1+\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}}}{1-\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}}}$$
$$= \frac{1+\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}}{1-\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}}$$
$$= \left(\frac{\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}}\right)\left(\frac{\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}}\right)$$
$$= \frac{2+\sqrt{3}+2-\sqrt{3}+2\sqrt{2+\sqrt{3}}\left(\sqrt{2-\sqrt{3}}\right)}{2+\sqrt{3}-\left(2-\sqrt{3}\right)}$$
$$= \frac{4+2\sqrt{4-3}}{2\sqrt{3}}$$
$$= \frac{3}{\sqrt{3}}$$
$$= \sqrt{3}$$

25. **ELECTRONICS** Consider an AC circuit consisting of a power supply and a resistor. If the current I_0 in the circuit at time t is $I_0 \sin t\theta$, then the power delivered to the resistor is $P = I_0^{\,2} R \sin^2 t\theta$, where R is the resistance. Express the power in terms of $\cos 2t\theta$.

SOLUTION:

The double angle formula is $\cos 2\theta = 1 - 2\sin^2\theta$.

$$2\sin^2\theta = 1 - \cos 2\theta$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Substitute $\frac{1-\cos 2t\theta}{2}$ for $\sin^2 t\theta$ in $P = I_0^{\,2} R \sin^2 t\theta$.

$$P = I_0^{\,2} R \sin^2 t\theta$$
$$P = I_0^{\,2} R\left(\frac{1-\cos 2t\theta}{2}\right)$$
$$P = \frac{1}{2} I_0^{\,2} R - \frac{1}{2} I_0^{\,2} R \cos 2t\theta$$

Verify that each equation is an identity.

26. $\tan 2\theta = \dfrac{2}{\cot\theta - \tan\theta}$

SOLUTION:

$$\tan 2\theta \stackrel{?}{=} \frac{2}{\cot\theta - \tan\theta}$$
$$\tan 2\theta \stackrel{?}{=} \frac{2}{\cot\theta - \tan\theta} \cdot \frac{\tan\theta}{\tan\theta}$$
$$\tan 2\theta \stackrel{?}{=} \frac{2\tan\theta}{\cot\theta\tan\theta - \tan^2\theta}$$
$$\tan 2\theta \stackrel{?}{=} \frac{2\tan\theta}{1 - \tan^2\theta}$$
$$\tan 2\theta = \tan 2\theta \checkmark$$

27. $1 + \dfrac{1}{2}\sin 2\theta = \dfrac{\sec\theta + \sin\theta}{\sec\theta}$

SOLUTION:

$$1 + \frac{1}{2}\sin 2\theta \stackrel{?}{=} \frac{\sec\theta + \sin\theta}{\sec\theta}$$
$$\stackrel{?}{=} \frac{\frac{1}{\cos\theta} + \sin\theta}{\frac{1}{\cos\theta}}$$
$$\stackrel{?}{=} \frac{\frac{1}{\cos\theta} + \sin\theta}{\frac{1}{\cos\theta}} \cdot \frac{\cos\theta}{\cos\theta}$$
$$\stackrel{?}{=} \frac{1}{\cos\theta} + \sin\theta(\cos\theta)$$
$$\stackrel{?}{=} 1 + \sin\theta\cos\theta$$
$$\stackrel{?}{=} 1 + \frac{2}{2}\sin\theta\cos\theta$$
$$\stackrel{?}{=} 1 + \frac{1}{2} \cdot 2\sin\theta\cos\theta$$
$$= 1 + \frac{1}{2}\sin 2\theta \checkmark$$

13-4 Double-Angle and Half-Angle Identities

28. $\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{\sin\theta}{2}$

SOLUTION:

$$\sin\frac{\theta}{2}\cos\frac{\theta}{2}\stackrel{?}{=}\frac{\sin\theta}{2}$$

$$\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2}\stackrel{?}{=}\frac{\sin\theta}{2}$$

$$\frac{\sin 2\left(\frac{\theta}{2}\right)}{2}\stackrel{?}{=}\frac{\sin\theta}{2}$$

$$\frac{\sin\theta}{2}=\frac{\sin\theta}{2}\checkmark$$

29. $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$

SOLUTION:

$$\tan\frac{\theta}{2}\stackrel{?}{=}\frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2}\stackrel{?}{=}\frac{\sin 2\left(\frac{\theta}{2}\right)}{1+\cos 2\left(\frac{\theta}{2}\right)}$$

$$\tan\frac{\theta}{2}\stackrel{?}{=}\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+2\cos^2\frac{\theta}{2}-1}$$

$$\tan\frac{\theta}{2}\stackrel{?}{=}\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$\tan\frac{\theta}{2}\stackrel{?}{=}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$\tan\frac{\theta}{2}=\tan\frac{\theta}{2}\checkmark$$

30. **FOOTBALL** Suppose a place kicker consistently kicks a football with an initial velocity of 95 feet per second. Prove that the horizontal distance the ball travels in the air will be the same for $\theta = 45° + A$ as for $\theta = 45° - A$. Use the formula given in Exercise 9.

SOLUTION:

$d = \frac{v^2 \sin 2\theta}{g}$.

For

$$\theta = 45° + A,$$
$$d = \frac{v^2 \sin 2(45° + A)}{g}$$
$$d = \frac{v^2 \sin(90° + 2A)}{g}$$
$$d = \frac{v^2 (\sin 90° \cos 2A + \cos 90° \sin 2A)}{g}$$
$$d = \frac{v^2 (1 \cdot \cos 2A + 0 \cdot \sin 2A)}{g}$$
$$d = \frac{v^2 \cos 2A}{g}$$

For

$$\theta = 45° - A,$$
$$d = \frac{v^2 \sin 2(45° - A)}{g}$$
$$d = \frac{v^2 \sin(90° - 2A)}{g}$$
$$d = \frac{v^2 (\sin 90° \cos 2A - \cos 90° \sin 2A)}{g}$$
$$d = \frac{v^2 (1 \cdot \cos 2A - 0 \cdot \sin 2A)}{g}$$
$$d = \frac{v^2 \cos 2A}{g}$$

Thus, the horizontal distance the ball travels in the air is the same.

Find the exact values of sin 2θ, cos2θ, and $\tan 2\theta$.

31. $\cos\theta = \frac{4}{5};\ 0° < \theta < 90°$

SOLUTION:

We know that, $\sin^2\theta = 1 - \cos^2\theta$.

Substitute $\frac{4}{5}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta &= 1-\left(\frac{4}{5}\right)^2\\ &= 1-\frac{16}{25}\\ &= \frac{9}{25}\\ \sin\theta &= \pm\frac{3}{5}\end{aligned}$$

Since $0° < \theta < 90°$, $\sin\theta = \frac{3}{5}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)\\ &= \frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{4}{5}\right)^2 - 1\\ &= 2\left(\frac{16}{25}\right) - 1\\ &= \frac{7}{25}\end{aligned}$$

Find $\tan\theta$.

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{3/5}{4/5}\\ &= \frac{3}{4}\end{aligned}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}$$
$$= \frac{\frac{6}{4}}{1-\frac{9}{16}}$$
$$= \frac{6}{\cancel{4}_1} \times \frac{\cancel{16}^4}{7}$$
$$= \frac{24}{7}$$

32. $\sin\theta = \frac{1}{3};\ 0 < \theta < \frac{\pi}{2}$

SOLUTION:

We know that $\cos^2\theta = 1 - \sin^2\theta$.

Substitute $\frac{1}{3}$ for $\sin\theta$.

$$\cos^2\theta = 1-\left(\frac{1}{3}\right)^2$$
$$= 1-\frac{1}{9}$$
$$= \frac{8}{9}$$
$$\cos\theta = \pm\frac{2\sqrt{2}}{3}$$

Since, $0 < \theta < \frac{\pi}{2}$, $\cos\theta = \frac{2\sqrt{2}}{3}$.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{1}{3}\right)\left(\frac{2\sqrt{2}}{3}\right)$$
$$= \frac{4\sqrt{2}}{9}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{2\sqrt{2}}{3}\right)^2 - 1\\ &= 2\left(\frac{8}{9}\right) - 1\\ &= \frac{16}{9} - 1\\ &= \frac{7}{9}\end{aligned}$$

Find $\tan\theta$.

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{1/3}{\frac{2\sqrt{2}}{3}}\\ &= \frac{1}{2\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(\frac{1}{2\sqrt{2}}\right)}{1-\left(\frac{1}{2\sqrt{2}}\right)^2}\\ &= \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{8}}\\ &= \frac{1}{\sqrt{2}} \times \frac{8}{7}\\ &= \frac{8}{7\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{4\sqrt{2}}{7}\end{aligned}$$

33. $\tan\theta = -3;\ 90° < \theta < 180°$

SOLUTION:

We know that $\tan^2\theta + 1 = \sec^2\theta$.
Substitute –3 for $\tan\theta$.

$$\begin{aligned}(-3)^2+1&=\sec^2\theta\\ 9+1&=\sec^2\theta\\ 10&=\sec^2\theta\\ \sec\theta&=\pm\sqrt{10}\\ \cos\theta&=\pm\frac{1}{\sqrt{10}}\end{aligned}$$

Since $90° < \theta < 180°$, $\cos\theta = -\frac{1}{\sqrt{10}}$.

We know that, $\sin^2\theta = 1-\cos^2\theta$.

Substitute $-\frac{1}{\sqrt{10}}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta&=1-\left(-\frac{1}{\sqrt{10}}\right)^2\\ &=1-\frac{1}{10}\\ &=\frac{9}{10}\\ \sin\theta&=\pm\frac{3}{\sqrt{10}}\end{aligned}$$

Since $90° < \theta < 180°$, $\sin\theta = \frac{3}{\sqrt{10}}$.

$$\begin{aligned}\sin 2\theta&=2\sin\theta\cos\theta\\ &=2\left(\frac{3}{\sqrt{10}}\right)\left(-\frac{1}{\sqrt{10}}\right)\\ &=-\frac{6}{10}\\ &=-\frac{3}{5}\end{aligned}$$

$$\begin{aligned}\cos 2\theta&=2\cos^2\theta-1\\ &=2\left(-\frac{1}{\sqrt{10}}\right)^2-1\\ &=2\left(\frac{1}{10}\right)-1\\ &=\frac{2}{10}-1\\ &=-\frac{4}{5}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2(-3)}{1-(-3)^2}\\ &= \frac{-6}{1-9}\\ &= \frac{3}{4}\end{aligned}$$

34. $\sec\theta = -\frac{4}{3};\ 90° < \theta < 180°$

SOLUTION:

$$\sec\theta = -\frac{4}{3}$$

$$\cos\theta = -\frac{3}{4}$$

We know that
$\sin^2\theta = 1-\cos^2\theta$.
Substitute $-\frac{3}{4}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta &= 1-\left(-\frac{3}{4}\right)^2\\ &= 1-\frac{9}{16}\\ &= \frac{7}{16}\\ \sin\theta &= \pm\frac{\sqrt{7}}{4}\end{aligned}$$

Since, $90° < \theta < 180°$, $\sin\theta = \frac{\sqrt{7}}{4}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{\sqrt{7}}{4}\right)\left(-\frac{3}{4}\right)\\ &= -\frac{3\sqrt{7}}{8}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{3}{4}\right)^2 - 1\\ &= 2\left(\frac{9}{16}\right) - 1\\ &= \frac{1}{8}\end{aligned}$$

Find $\tan\theta$.

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{\sqrt{7}/4}{-3/4}\\ &= -\frac{\sqrt{7}}{3}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(-\frac{\sqrt{7}}{3}\right)}{1-\left(-\frac{\sqrt{7}}{3}\right)^2}\\ &= \frac{-\frac{2\sqrt{7}}{3}}{1-\frac{7}{9}}\\ &= -\frac{\cancel{2}\sqrt{7}}{\cancel{3}} \times \frac{\overset{3}{\cancel{9}}}{\cancel{2}}\\ &= -3\sqrt{7}\end{aligned}$$

35. $\csc\theta = -\frac{5}{2};\ \frac{3\pi}{2} < \theta < 2\pi$

SOLUTION:

$$\csc\theta = -\frac{5}{2}$$
$$\sin\theta = -\frac{2}{5}$$

We know that
$\cos^2\theta = 1 - \sin^2\theta$.
Substitute $-\frac{2}{5}$ for $\sin\theta$.

$$\cos^2\theta = 1 - \left(-\frac{2}{5}\right)^2$$
$$= 1 - \frac{4}{25}$$
$$= \frac{21}{25}$$
$$\cos\theta = \pm\frac{\sqrt{21}}{5}$$

Since $\frac{3\pi}{2} < \theta < 2\pi$, $\cos\theta = \frac{\sqrt{21}}{5}$.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(-\frac{2}{5}\right)\left(\frac{\sqrt{21}}{5}\right)$$
$$= -\frac{4\sqrt{21}}{25}$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$= 2\left(\frac{\sqrt{21}}{5}\right)^2 - 1$$
$$= 2\left(\frac{21}{25}\right) - 1$$
$$= \frac{42}{25} - 1$$
$$= \frac{17}{25}$$

Find $\tan\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$= \frac{-2/5}{\sqrt{21}/5}$$
$$= \frac{-2}{\sqrt{21}}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= \frac{2\left(\frac{-2}{\sqrt{21}}\right)}{1-\left(\frac{-2}{\sqrt{21}}\right)^2}$$
$$= \frac{\frac{-4}{\sqrt{21}}}{1-\frac{4}{21}}$$
$$= \frac{-4}{\sqrt{21}} \times \frac{21}{17}$$
$$= -\frac{4\sqrt{21}}{17}$$

36. $\cot\theta = \frac{3}{2};\ 180° < \theta < 270°$

SOLUTION:

$$\cot\theta = \frac{3}{2}$$
$$\tan\theta = \frac{2}{3}$$

We know that
$\tan^2\theta + 1 = \sec^2\theta$.
Substitute $\frac{2}{3}$ for $\tan\theta$.

$$\left(\frac{2}{3}\right)^2 + 1 = \sec^2\theta$$
$$\frac{4}{9} + 1 = \sec^2\theta$$
$$\frac{13}{9} = \sec^2\theta$$
$$\sec\theta = \pm\frac{\sqrt{13}}{3}$$
$$\cos\theta = \pm\frac{3}{\sqrt{13}}$$

Since, $180° < \theta < 270°$, $\cos\theta = -\frac{3}{\sqrt{13}}$.
We know that
$\sin^2\theta = 1 - \cos^2\theta$.

Substitute $-\frac{3}{\sqrt{13}}$ for $\cos\theta$.

$$\begin{aligned}\sin^2\theta &= 1-\left(-\frac{3}{\sqrt{13}}\right)^2\\ &= 1-\frac{9}{13}\\ &= \frac{4}{13}\\ \sin\theta &= \pm\frac{2}{\sqrt{13}}\end{aligned}$$

Since, $180° < \theta < 270°$, $\sin\theta = -\frac{2}{\sqrt{13}}$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{2}{\sqrt{13}}\right)\left(-\frac{3}{\sqrt{13}}\right)\\ &= \frac{12}{13}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{3}{\sqrt{13}}\right)^2 - 1\\ &= 2\left(\frac{9}{13}\right) - 1\\ &= \frac{18}{13} - 1\\ &= \frac{5}{13}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)^2}\\ &= \frac{\frac{4}{3}}{1-\frac{4}{9}}\\ &= \frac{4}{3}\times\frac{9}{5}\\ &= \frac{12}{5}\end{aligned}$$

37. **CCSS CRITIQUE** Teresa and Nathan are calculating the exact value of sin 15°. Is either of them correct? Explain your reasoning.

Teresa

$$\begin{aligned}\sin(A-B) &= \sin A\cos B - \cos A\sin B\\ \sin(45-30) &= \sin 45\cos 30 - \cos 45\sin 30\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{4}}{4}\end{aligned}$$

Nathan

$$\begin{aligned}\sin\frac{A}{2} &= \pm\sqrt{\frac{1-\cos A}{2}}\\ \sin\frac{30}{2} &= \pm\sqrt{\frac{1-\frac{1}{2}}{2}}\\ &= 0.5\end{aligned}$$

SOLUTION:

$$\begin{aligned}\sin 15° &= \sin(45° - 30°)\\ &= \sin 45° \cos 30° - \cos 45° \sin 30°\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

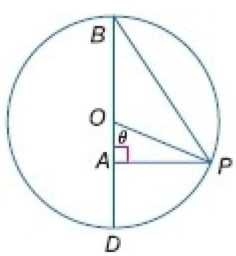
$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\sin 15° = \sin\frac{30°}{2}$$

$$\begin{aligned}\sin\frac{30°}{2} &= \pm\sqrt{\frac{1-\cos 30°}{2}}\\ &= \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \pm\sqrt{\frac{2-\sqrt{3}}{4}}\end{aligned}$$

No; Teresa incorrectly added the square roots, and Nathan used the half-angle identity incorrectly. He used sin 30° in the formula instead of first finding the cosine.

38. **CHALLENGE** Circle *O* is the unit circle. Use the figure to prove that $\tan\frac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta}$.



SOLUTION:

$\angle PBD$ is an inscribed angle that subtends the same arc as the central angle $\angle POD$, so $m\angle POD = \frac{1}{2}\theta$.

By right triangle trigonometry, $\tan\frac{1}{2}\theta = \frac{PA}{BA} = \frac{PA}{1+OA} = \frac{\sin\theta}{1+\cos\theta}$.

39. **WRITING IN MATH** Write a short paragraph about the conditions under which you would use each of the three identities for cos 2θ.

SOLUTION:

If you are only given the value of cos θ, then $\cos 2\theta = 2\cos^2\theta - 1$ is the best identity to use. If you are only given the value of sin θ, then $\cos 2\theta = 1 - 2\sin^2\theta$ is the best identity to use. If you are given the values of both cos θ and sin θ, then $\cos 2\theta = \cos^2\theta - \sin^2\theta$ works just as well as the other two.

40. **PROOF** Use the formula for sin $(A + B)$ to derive the formula for sin 2θ, and use the formula for cos $(A + B)$ to derive the formula for cos 2θ.

SOLUTION:

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta)\\ &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

You can find alternate forms for $\cos 2\theta$ by making substitutions into the expression $\cos^2\theta - \sin^2\theta$.

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= (1 - \sin^2\theta) - \sin^2\theta\\ &= 1 - 2\sin^2\theta\end{aligned}$$

Substitute $1 - \sin^2\theta$ for $\cos^2\theta$.

Simplify.

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= \cos^2\theta - (1 - \cos^2\theta)\\ &= 2\cos^2\theta - 1\end{aligned}$$

Substitute $1 - \cos^2\theta$ for $\sin^2\theta$.
Simplify.

13-4 Double-Angle and Half-Angle Identities

41. **REASONING** Derive the half-angle identities from the double-angle identities.

SOLUTION:

$1-2\sin^2\theta=\cos 2\theta$	Double-angle identity
$1-2\sin^2\frac{A}{2}=\cos A$	Substitute $\frac{A}{2}$ for θ and A for 2θ
$\sin^2\frac{A}{2}=\frac{1-\cos A}{2}$	Solve for $\sin^2\frac{A}{2}$
$\sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}}$	Take the square root of each side.

Find $\cos\frac{A}{2}$.

$2\cos^2\theta-1=\cos 2\theta$	Double-angle identity
$2\cos^2\frac{A}{2}-1=\cos A$	Substitute $\frac{A}{2}$ for θ and A for 2θ
$\cos^2\frac{A}{2}=\frac{1+\cos A}{2}$	Solve for $\cos^2\frac{A}{2}$
$\cos\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{2}}$	Take the square root of each side.

42. **OPEN ENDED** Suppose a golfer consistently hits the ball so that it leaves the tee with an initial velocity of 115 feet per second and $d=\frac{v^2\sin 2\theta}{g}$. Explain why the maximum distance is attained when $\theta=45°$.

SOLUTION:

Sample answer: Since $d=\frac{v^2\sin 2\theta}{g}$, *d* is at a maximum when $\sin 2\theta=1$, that is, when $2\theta=90°$ or $\theta=45°$.

43. **SHORT RESPONSE** Angles *C* and *D* are supplementary. The measure of angle *C* is seven times the measure of angle *D*. Find the measure of angle *D* in degrees.

SOLUTION:

$$m\angle C+m\angle D=180°$$
$$m\angle C=7m\angle D$$
$$C+D=180 \quad \text{Solve system.}$$
$$C=7D$$
$$7D+D=180$$
$$8D=180$$
$$D=22.5$$

The measure of angle D is 22.5 degrees.

44. **SAT/ACT** Ms. Romero has a list of the yearly salaries of the staff members in her department. Which measure of data describes the middle income value of the salaries?

A mean
B median
C mode
D range
E standard deviation

SOLUTION:

The median represents the middle value of a set of data. So, the correct choice is B.

45. Identify the domain and range of the function $f(x) = |4x + 1| - 8$.

F. $D = \{x | -3 \le x \le 1|\}, R = \{y | y \ge -8\}$.

G. $D = \{\text{all real numbers}\}, R = \{y | y \ge -8\}$

H. $D = \{x | -3 \le x \le 1|\}, R = \{\text{all real numbers}\}$

J.
$D = \{\text{all real numbers}\}, R = \{\text{all real numbers}\}$

SOLUTION:

$f(x) = |4x + 1| - 8$.

Domain = $D = \{\text{all real numbers}\}$

Range = $R = \{y | y \ge -8\}$

So, the correct choice is G.

46. **GEOMETRY** Angel is putting a stone walkway around a circular pond. He has enough stones to make a walkway 144 feet long. If he uses all of the stones to surround the pond, what is the radius of the pond?

A. $\frac{144}{\pi}$ ft

B. $\frac{72}{\pi}$ ft

C. 144π ft

D. 72π ft

SOLUTION:

The circumference of the pond is 144 ft.

$$\begin{aligned} 2\pi r &= 144 \\ r &= \frac{144}{2\pi} \\ r &= \frac{72}{\pi} \end{aligned}$$

So, the radius of the pond is $\frac{72}{\pi}$ ft, and the correct choice is B.

Find the exact value of each expression.

47. $\sin 135°$

SOLUTION:

$$\begin{aligned}\sin 135° &= \sin(180° - 45°)\\ &= \sin 180° \cos 45° - \cos 180° \sin 45°\\ &= 0 \cdot \frac{1}{\sqrt{2}} - (-1) \cdot \frac{1}{\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

48. $\cos 105°$

SOLUTION:

$$\begin{aligned}\cos 105° &= \cos(60° + 45°)\\ &= \cos 60° \cos 45° - \sin 60° \sin 45°\\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}\\ &= \frac{1 - \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

13-4 Double-Angle and Half-Angle Identities

49. $\sin 285°$

SOLUTION:

$$\begin{aligned}\sin 285° &= \sin(60° + 225°)\\ &= \sin 60° \cos 225° + \cos 60° \sin 225°\\ &= \frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\cdot\left(-\frac{1}{\sqrt{2}}\right)\\ &= -\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\\ &= -\frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{-\sqrt{3}-1}{2\sqrt{2}} \times \frac{2\sqrt{2}}{2\sqrt{2}}\\ &= \frac{2\sqrt{2}\left(-\sqrt{3}-1\right)}{2\sqrt{2}\left(2\sqrt{2}\right)}\\ &= \frac{2\left(-\sqrt{6}-\sqrt{2}\right)}{8}\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

50. $\cos(-30°)$

SOLUTION:

Since $\cos(-\theta) = \cos\theta$, $\cos(-30°) = \cos(30°)$.

$$\cos(30°) = \frac{\sqrt{3}}{2}$$

51. $\sin(-240°)$

SOLUTION:

Since $\sin(-\theta) = -\sin\theta$, $\sin(-240°) = -\sin 240°$.

$$\begin{aligned}-\sin 240° &= -\sin(60° + 180°)\\ &= -\sin 60° \cos 180° + \cos 60° \sin 180°\\ &= -\frac{\sqrt{3}}{2}\cdot(-1) + \frac{1}{2}\cdot(0)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

52. $\cos(-120°)$

SOLUTION:

Since $\cos(-\theta) = \cos\theta$,

$$\begin{aligned}\cos(-120°) &= \cos(120°) \\ &= \cos(60° + 60°) \\ &= \cos 60° \cos 60° - \sin 60° \sin 60° \\ &= \frac{1}{2} \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{1}{4} - \frac{3}{4} \\ &= -\frac{2}{4} \\ &= -\frac{1}{2}\end{aligned}$$

Verify that each equation is an identity.

53. $\cot\theta + \sec\theta = \dfrac{\cos^2\theta + \sin\theta}{\sin\theta\cos\theta}$

SOLUTION:

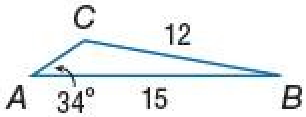
$$\begin{aligned}\cot\theta + \sec\theta &\stackrel{?}{=} \frac{\cos^2\theta + \sin\theta}{\sin\theta\cos\theta} \\ \cot\theta + \sec\theta &\stackrel{?}{=} \frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin\theta}{\sin\theta\cos\theta} \\ \cot\theta + \sec\theta &\stackrel{?}{=} \frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta} \\ \cot\theta + \sec\theta &= \cot\theta + \sec\theta \checkmark\end{aligned}$$

54. $\sin^2\theta + \tan^2\theta = (1 - \cos^2\theta) + \dfrac{\sec^2\theta}{\csc^2\theta}$

SOLUTION:

$$\begin{aligned}\sin^2\theta + \tan^2\theta &\stackrel{?}{=} (1 - \cos^2\theta) + \frac{\sec^2\theta}{\csc^2\theta} \\ \sin^2\theta + \tan^2\theta &\stackrel{?}{=} \sin^2\theta + \frac{\sec^2\theta}{\csc^2\theta} \\ \sin^2\theta + \tan^2\theta &\stackrel{?}{=} \sin^2\theta + \frac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}} \\ \sin^2\theta + \tan^2\theta &\stackrel{?}{=} \sin^2\theta + \frac{\sin^2\theta}{\cos^2\theta} \\ \sin^2\theta + \tan^2\theta &= \sin^2\theta + \tan^2\theta \checkmark\end{aligned}$$

Determine whether each triangle should be solved by beginning with the Law of *Sines* or Law of *Cosines*. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

55. 

SOLUTION:

We are given two sides and a non-included angle, which means there are two possible solutions using the law of sines.

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$\frac{\sin 34°}{12} = \frac{\sin C}{15}$$
$$\sin C = \frac{15\sin 34°}{12}$$
$$C = \sin^{-1}\left(\frac{15\sin 34°}{12}\right)$$
$$C = 44° \text{ or } 136°$$

Solution 1: $m\angle C = 44°$

$m\angle B = 180° - 34° - 44° = 102°$

Now that we have two sides and an included angle, we can use the law of cosines to solve for b.

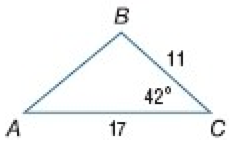
$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$
$$b^2 = 15^2 + 12^2 - 2(15)(12)\cos 102°$$
$$b = \sqrt{15^2 + 12^2 - 2(15)(12)\cos 102°}$$
$$b = 21.0$$

Solution 2: $m\angle C = 136°$

$m\angle B = 180° - 34° - 136° = 10°$

Now that we have two sides and an included angle, we can use the law of cosines to solve for b.

$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$
$$b^2 = 15^2 + 12^2 - 2(15)(12)\cos 10°$$
$$b = \sqrt{15^2 + 12^2 - 2(15)(12)\cos 10°}$$
$$b = 3.9$$

56. 

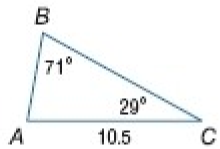
SOLUTION:

The triangle should be solved by beginning with the Law of Cosines.

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 11^2 + 17^2 - 2(11)(17)\cos 42°$$
$$c^2 \approx 132.06$$
$$c \approx 11.5$$

Use the Law of Sines to find a missing angle measure.

$$\frac{\sin A}{11} \approx \frac{\sin 42°}{11.5}$$
$$\sin A \approx \frac{11\sin 42°}{11.5}$$
$$\sin A \approx 0.6400$$
$$A \approx 40°$$
$$m\angle B \approx 180° - (40° + 42°)$$
$$\approx 98°$$

57. 

SOLUTION:

The triangle should be solved by beginning with the Law of Sines.

$$m\angle A = 180° - (71° + 29°)$$
$$= 80°$$

Use Law of Sines to find a.

$$\frac{\sin 80°}{a} = \frac{\sin 71°}{10.5}$$
$$a = \frac{10.5 \sin 80°}{\sin 71°}$$
$$a \approx 10.9$$

Use Law of Sines to find c.

$$\frac{\sin 71°}{10.5} = \frac{\sin 29°}{c}$$
$$c = \frac{10.5 \sin 29°}{\sin 71°}$$
$$c \approx 5.4$$

Solve each equation by factoring

58. $x^2 + 5x - 24 = 0$

SOLUTION:

$$x^2 + 5x - 24 = 0$$
$$x^2 + 8x - 3x - 24 = 0$$
$$x(x+8) - 3(x+8) = 0$$
$$(x+8)(x-3) = 0$$

$x + 8 = 0$; $x - 3 = 0$
$x = -8$; $x = 3$

The solution set is {–8, 3}.

59. $x^2 - 3x - 28 = 0$

SOLUTION:

$$\begin{aligned} x^2 - 3x - 28 &= 0 \\ x^2 - 7x + 4x - 28 &= 0 \\ x(x-7) + 4(x-7) &= 0 \\ (x-7)(x+4) &= 0 \end{aligned}$$

$x - 7 = 0$; $x + 4 = 0$
$x = 7$; $x = -4$

The solution set is $\{-4, 7\}$.

60. $x^2 - 4x = 21$

SOLUTION:

$$\begin{aligned} x^2 - 4x &= 21 \\ x^2 - 4x - 21 &= 0 \\ x^2 - 7x + 3x - 21 &= 0 \\ x(x-7) + 3(x-7) &= 0 \\ (x-7)(x+3) &= 0 \end{aligned}$$

$x - 7 = 0$; $x + 3 = 0$
$x = 7$; $x = -3$

The solution set is $\{-3, 7\}$.